the eight solutions give one exact value of the frequency determinant, the particular solution in Stodola's method is for only one iteration that must be repeated to achieve convergence. Therefore, the number of particular solutions needed in Stodola's method to obtain one natural frequency clearly depends on how closely the assumed solution matches the exact solution, and, contrary to the statement of Ref. 1, it seems to the writer quite impossible to say which one of the two methods will require a smaller number of solutions per one natural frequency.

Beyond the question of shorter execution time, the method presented in Ref. 1 has several serious disadvantages that should be considered before it is used. First of all, the method is capable of finding only the lowest mode plus a certain number of successive modes for each circumferential wave number. Exactly how many, is not clear. In Ref. 1 it is stated that, because of accumulation of errors in successive modes and limited computer storage, about ten modes may be obtained, although not more than three for each case are displayed in the paper. It would seem to the writer that the number of modes which can be obtained with the method of Ref. 1 again is related directly to the choice of the assumed solutions. This can be concluded from the detailed calculations presented in Ref. 3 on p. 205, where it is shown that if a poor initial guess of the displacements is made, then, because of accumulation of errors at each iteration, difficulties may arise already in the second mode.

Second, the method of Ref. 1 always requires the calculation of all the preceding modes, down to the fundamental mode, before a particular desired mode of free vibration can be obtained. This means that whenever it is required to examine a given frequency interval, it is necessary to calculate all the preceding modes, regardless of whether or not they are wanted.

The incapability to find the natural frequencies in a given frequency interval, without calculating the preceding modes and the limited number of modes that can be obtained, is perhaps the most serious disadvantage of the method of Ref. 1. For many applications, such as in the construction of a transient solution, more than ten modes are needed. For example, for a hemispherical shell subjected to a sudden pressure change, about eighteen modes are required in the transient solution.[4] For other applications, such as the determination of resonant frequencies of a shell of revolution within a frequency interval, the method would fail completely if the resonant frequencies are among those modes that are destroyed by accumulative errors.

In addition to these two severe limitations of the method of Ref. 1, there is another feature inherent in Stodola's method which could become a serious problem when the method is applied to free vibration of shells of revolution. According to Ref. 1, one has to provide intelligent guesses for four displacement functions over the complete meridian of the shell for every mode of free vibration. For a beam, where only one displacement function is needed, this may not be too difficult. For some simple shells, especially for which the exact solutions are known a priori, this may not be much of a problem either. It should be kept in mind, however, that the method is claimed to be "automatic" for an arbitrary shell of revolution. It seems to the writer that the procedure of guessing four different functions, which ought to resemble the actual mode, for each natural frequency can be rather difficult and make the method certainly somewhat less than "automatic."

By contrast, the multisegment method is capable of calculating any one or any number of natural frequencies, and all relevant variables of a mode within any frequency interval. The calculated values of the frequency equation can be made as accurate as desired. The calculation is straightforward and involves no guessing of any kind.

Finally, as a subscriber and reader of the AIAA Journal, the writer is somewhat concerned about the wasted space in Ref. 1 devoted to the fact that in one of his earlier papers in place of mode #3 he inadvertently inserted mode #4. Thus, the first three bending modes were included, but the first membrane mode, which actually is below the third bending mode, was omitted. The author of Ref. 1 apparently has tried to make the most of this bookkeeping error as if the sole contribution of this paper would be the discovery of mode 3 of a 60° spherical shell with a thickness-to-radius ratio of 0.05.

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Reply by Author to A. Kalnins

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IN reply to Kalnins' comments on the method of shell vibration analysis presented in Ref. 1, I should first of all like to clarify some points of semantics. He uses the term "multisegment method" to describe the method presented in Ref. 2. It should be pointed out that both methods, being based on the technique of forward integration, require shell segmentation in order to avoid round-off errors associated with the rapid growth of fundamental solutions. Therefore, in the following, the method in which the input to a given iteration step is an estimate of the displacement modes[1] is called the mode method, and the method in which the input to a given iteration step is an estimate of the frequency[2] is called the frequency method. I agree that the frequency method is not an iterative method in the same sense as the mode method. As Kalnins himself describes it, the frequency method is essentially a systematized process of trial and error, whereas the mode method is a true iteration method with proven convergence. However, for the purpose of discussion, here, as in Ref. 1, each determinant evaluation of the frequency method is referred to as an iteration step.

With the preceding in mind, I should like to correct Kalnins' misinterpretation of the following statement of Ref. 1. "Thus, other things being equal, one can expect a considerably shorter execution time per step in the mode method." When read in context, it is clear that the word "step" refers to a single iteration step and not the complete solution for a given vibration mode.

Kalnins' alleged disadvantages of the mode method may be summarized, for convenience, to be the following: 1) The number of iterations required to converge to a given mode depends on "intelligent" initial guesses for the four modal displacement functions. 2) The method is capable of finding only the first few lowest modes for each circumferential wave number. 3) No mode can be obtained without obtaining previously all the preceding lower modes with the same number of circumferential waves.

The refutation of these statements follows: 1) In the mode method no attempt need be made to speed up convergence by providing intelligent initial guesses for the displacement modes. In fact, for all modes the initial guess built into our program is simply $u = v = \chi = 0$ and $w =$ linear function of s. As a result, the rate of convergence de-

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pends primarily on the frequency spacing, and a good rule-of-thumb for the number of iterations n_p required to converge to the p-th mode with frequency ω_p is

$$n_p \sim N/4 \log_{10}(\omega_{p+1}/\omega_p)$$

where N is the number of significant digits desired in the frequency. For example, in the third example of Ref. 1, giving vibration modes of a quite general composite shell, the number of iterations required to obtain frequencies to four digit accuracy was $n_1 = 4$, $n_2 = 11$ for the axisymmetric modes and $n_1 = 5$, $n_2 = 4$ for the first harmonic modes. Furthermore, my experience with a variety of other shells is that the number of iterations required for convergence is relatively insensitive to the initial guess.

2) It was stated in Ref. 1 that ten modes can be obtained and no tape storage is used. The number ten was chosen arbitrarily for the purpose of dimensioning certain quantities in the program. In fact, there are 3600 unused locations in the 32 K core, which if employed, at a rate of 400 locations per mode could store nine additional modes, if desired. However, as explained below, it is doubtful that storage of more than nine modes currently retained will ever be necessary. It was also stated in Ref. 1 that no significant deterioration of accuracy has been observed for the higher modes. The reason for this is simply that, in contrast to the usual laborious hand computation for higher modes of beams using Stodola's method, the orthogonalization with respect to lower modes is made after each iteration; furthermore, this calculation is based on an "inner product" subroutine which is generally accurate to five or six significant digits, considerably more significance than required in modes themselves.

In addition, the method can be modified slightly so that the calculation for a given mode can be made not to depend on previous modes, thereby eliminating any practical limit on the number of modes obtainable. In the mode method an approximation for the next higher eigenvalue (i.e. ω^2) is readily available by virtue of a property of the Rayleigh quotient; namely, the ratio of successive differences of the sequence of square frequency estimates given by Eq. (27) of Ref. 1 converges to the fourth power of the ratio of the next higher frequency to the one being obtained. By the amount of this approximation, the eigenvalue spectrum may be shifted to make the next desired eigenvalue the minimum (in absolute value) of the modified system of equations. As a result, convergence to the corresponding mode no longer depends on the orthogonalization with respect to previously obtained modes. It is also noted that the rate of convergence after such a shift is greatly improved, since the frequency ratio is enlarged by virtue of the desired frequency being now close to zero†. The trade-off in applying such a procedure is that after the eigenvalue shift the differential operator is altered, thereby requiring one recomputation of the complementary solutions (as is necessary in each step of the frequency method). Since roughly five to eight iterations can be made in the time required to compute the complementary solutions, the desirability of applying this procedure for each mode is not clear-cut.

In practice, however, as a result of applying the shifting procedure for very high modes, orthogonalization with respect to only the previous nine modes is probably sufficient to obtain an arbitrary number of modes. It is only necessary that after the shift the next two desired modes have eigenvalues smaller in absolute value than the shifted eigenvalues of previous modes with respect to which no orthogonalization is made. It should be clear from the foregoing that the mode method is not limited in the number of modes obtainable.

3) Since an application of an eigenvalue shift can make any desired mode have the minimum eigenvalue of the modified system of equations, it is clearly unnecessary to obtain all the lower modes of the unmodified system prior to obtaining the desired mode. For example, if a given frequency interval is to be examined, one can shift the middle of this interval to the zero of the frequency axis.

Finally, in reference to Kalnins' last comment, it should be mentioned that the purpose of the first example of Ref. 1 was simply to illustrate the application of the mode method to a shell which had been previously studied in detail by use of Legendre functions[3] and by the frequency method.[2] Since in Ref. 3 seven frequencies are presented for this shell and in Ref. 2 three frequencies are presented, none of which contain the third frequency presented in Ref. 1, the natural conclusion was that the third mode had apparently been overlooked.

† This technique was brought to my attention by G. A. Greenbaum in a private communication.

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Comments on "Effect of Microparticle Impact on the Optical Properties of Metals"

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IN the introduction as well as in other sections of Ref. 1, the authors have stated that the motivation for their efforts was the lack of information regarding "the effect of micrometeoroid bombardment on the surface finish of a space vehicle." Hence they felt the need "to simulate the effect of micrometeoroid impacts on candidate skin materials and to determine the resulting change in optical properties." Although Ref. 1 is an adequate description of a multifaceted study, it is unfortunate indeed that the authors were unaware, apparently, of the work of Refs. 2–5. For, if they had been aware of it, they could have presented the results they obtained in a more meaningful fashion, and they would have immediately understood that the projectiles they used in their experiments (100-μ diam W particle) are inappropriate for the purpose intended. Such particles can hardly be used to determine damage to the optical properties of satellite skin materials by micrometeoroids in space since by the highest flux estimates of Refs. 5 and 6, it would require $\sim 10^4$ yr for just one such particle to strike an area the size of the disc used in their experiments (~ 1 cm^2). It is possible to use particles somewhat different from those causing the surface damage in space ($<10^{-10}$g); however, extrapolating the effects of laboratory exposures to the actual degradation of surface reflectance caused by microparticles in space at least involves the scaling laws presented and discussed in Ref. 4, and is not a matter of simply presenting measured reflectances.

Although very carefully presented by the authors as observed results only, the reflectance data are used to point out that high velocity impacts caused more degradation than did low velocity impacts. This conclusion is really not too helpful, for the simplest possible analysis would show that

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